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MATHEMATICAL PROGRAMMING IN OPTIMAL PLASTIC DESIGN

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Abstract—The paper is intended to discuss problems of optimal plastic design of structures from the view point of mathematical programming. Existing methods of optimal design are shown to be corresponding to various formulations of mathematical programs. The duality theorems of mathematical programming can then be used to obtain necessary and sufficient criteria of optimum. Programming as a method of numerical solution is also indicated.

1. INTRODUCTION

OPTIMAL plastic design of structures is concerned with obtaining a structure of minimum "cost" which, for a specified yield condition of the material, safely equilibrates given loadings. The first major contribution to this subject was given by Michell [1], who obtained a sufficient condition for pinjointed frameworks to be of minimum weight. Foulkes [2] generalised Michell's result to portal frames. Drucker and Shield [3] obtained the general sufficient condition of minimum weight for other types of structures. In these cases, the cost function is linear with respect to the design variables. Prager [4], Vargo [5] and Shield [6] considered problems where the cost function is nonlinear. More recently, a general theory of plastic design for a convex cost function is given by Marçal and Prager [7] and Prager and Shield [8].

Solutions of specific design problems are few and are obtained mostly analytically for the sake of illustrating the theory, except perhaps in the cases of minimum weight design of frameworks and portal frames. These latter cases owe their success to a large extent to the application of linear programming methods. Such methods have been noticed and applied by a number of authors, for example Foulkes [2], Livesley [9], Hemp [10]; Prager [11]. However, the theories of non-linear programming are still not often used. It is the purpose of this paper to exploit the applicability of mathematical programming to optimal plastic design. Some general statements may be given as follows:

(i) An unified view of optimal design is obtained by formulating the problem as a programming problem.
(ii) Programming offers a feasible means of obtaining numerical solutions or obtaining bounds to the cost of the optimal structure.
(iii) The numerical solutions obtained are often close approximations to the theoretical optimum and may give indications of how to solve the problems analytically.
(iv) Necessary and sufficient condition for a design to be an optimum can be derived by using the duality theorems of mathematical programming.

Some of the above statements have already been recognised in previous works. In the following sections, various design problems will be given to illustrate them. To begin with, previous works along these directions are now reviewed. Foulkes [2] studied minimum weight design of portal frames in great detail and obtained all possible results of (ii)–(iv). Gross and Prager [12] used linear programming and the idea of (iii) to obtain the minimum weight design of a beam subjected to moving loads. Dorn *et al.* [13] applied linear programming to framework design but did not relate their result to the general theory of plastic design. This may be due to the fact that an optimum framework is statically determinate and is therefore an elastic design as well. Hemp [10] and Chan [14] showed that, for pin-jointed frameworks, a necessary and sufficient condition for minimum weight can be derived from the duality theorem of linear programming, which is in fact Michell's original result. Extension of this result to alternative loading cases is also obtained in Hemp [10] and Chan [15].

A main purpose of this paper is to extend these results to non-linear cost functions. With emphasis on the numerical method of solution, the original continuous design problems are first turned into optimization problems with discrete variables. So that besides its own theoretical interest, this paper may be viewed as a computational study of the existing theories of optimum design.

Some results in mathematical programming will be summarized in the following section in a form suitable for further uses.

2. MATHEMATICAL PROGRAMMING

Consider the problem of

(I) $$\text{Minimize } g(x) = (x_1, \ldots, x_n),$$

subject to

$$G_i(x) \leq 0, \qquad i = 1, \ldots, m;$$

where the functions g, G_i have continuous derivatives for all x_j.

Construct the Lagrangian function

$$U(x, y) = g(x) + \sum_{i=1}^{m} y_i G_i(x),$$

where

$$y = (y_1, \ldots, y_m).$$

The following are special cases of the theorems first studied by Kuhn and Tucker [16];

THEOREM 1. *In order that $\bar{x}$ be a solution of problem* I, *it is necessary that $\bar{x}$ and some $\bar{y}$ satisfy*

(1) $$\frac{\partial U(\bar{x}, \bar{y})}{\partial x_j} = 0, \qquad j = 1 \ldots, n;$$

(2) $$G_i(\bar{x}) \leq 0, \qquad \bar{y}_i G_i(\bar{x}) = 0, \qquad \bar{y}_i \geq 0, \qquad i = 1, \ldots, m.$$

Remark. The proof of this result in general will require additional assumptions, (see Hadley [17], section 6–2/3). In case $G_i(x)$ are all linear functions of x, the theorem is given implicitly in Dorn [18].

THEOREM 2. *In order that* $\bar{x}$ *be a solution of problem* I, *it is sufficient that* $\bar{x}$ *and some* $\bar{y}$ *satisfy conditions* (1), (2) *and*

$$U(\bar{x}, \bar{y}) \leq U(x, \bar{y}) \tag{3}$$

for all x satisfying $G_i(x) \leq 0$, *all i.*

Remark. The proof of this result follows the same argument as theorem 2 of [16].

The dual problem of problem I will now be defined as

$$\text{Maximize } U(x, y) = g(x) + \sum_{i=1}^{m} y_i G_i(x), \tag{II}$$

subject to

$$\frac{\partial g}{\partial x_j} + \sum_{i=1}^{m} y_i \frac{\partial G_i}{\partial x_j} = 0, \qquad j = 1, \ldots, n;$$

$$y_i \geq 0, \qquad i = 1, \ldots, m.$$

The following results holds for the above pair of problems:

Duality Theorem (Dorn [18])

D1. *Suppose problem* I *has an optimal solution* $\bar{x}$, *then there exists an* $\bar{y}$ *such that* $(\bar{x}, \bar{y})$ *is an optimal solution of problem* II *with the property* $g(\bar{x}) = U(\bar{x}, \bar{y})$ *if the following conditions are satisfied:*

(4) $g(x)$ *is a convex function of* x;

(5) $G_i(x)$ *are all linear functions of* x.

D2. *Suppose problem* II *has an optimal solution* $(\bar{x}, \bar{y})$, *then* $\bar{x}$ *is an optimal solution of problem* I *with the property* $g(\bar{x}) = U(\bar{x}, \bar{y})$ *if the following conditions are satisfied:*

(6) $g(x)$ *is either a linear or a strictly convex function of* x;

(7) $G_i(x)$ *are all linear functions of* x.

Remark. It should be noted that an optimal solution $(\bar{x}, \bar{y})$ of problems I, II will satisfy the conditions (1), (2).

The only problems examined in this paper are those for which $G_i(x)$ are all linear. However, the theorems of this section are deliberately written in such a form as to show where this linearity requirement is needed. Additional conditions on the functions g, G_i would be required if the functions are not all linear, see for example [16]. These theorems will be applied in the following sections to optimal design problems. It may be worth noting that problem I can include equality constraints $G_i(x) = 0$ by writing $G_i(x) \leq 0$, $-G_i(x) \leq 0$.

3. OPTIMAL DESIGN OF BEAMS FOR GIVEN LOADS

Optimal design of beams subjected to given loads may be formulated statically as

$$\text{Minimize } C = \int_0^l \varphi(R)\, dx,$$

subject to

$$f(M) \leq R,$$

$$\frac{d^2M}{dx^2} = -p(x);$$

where x is measured along a beam of length l, φ is the cost function of the plastic resistance $R(x)$ at every cross-section and f is a convex yield function of the bending moment $M(x)$ which satisfies the equation of equilibrium for the given loading $p(x)$. Heyman [19] considered the case where $\varphi(R) = R$ and $f(M) = |M|$. Marçal and Prager [7] treated a more general case where $\varphi(R)$ is a convex function of R and $f(M) = \max\{|M|, R_0\}$, R_0 = constant being the required minimum plastic resistance of the structure.

In order to treat the above problem by mathematical programming, the integral and derivative are approximated by finite sum and differences, and the loadings are represented by concentrated loads in the intervals $0 < x_1 < x_2 < \ldots < x_n = l$. The problem is then reformulated as

$$\text{(8)} \qquad \text{Minimize } C = \sum_{j=1}^{n} c_j \varphi(R_j),$$

subject to

$$f(M_j) \leq R_j, \qquad j = 1, \ldots, n;$$

$$\sum_{j=1}^{n} a_{ij} M_j = p_i, \qquad i = 1, \ldots, m;$$

where $n-m > 0$ for statically indeterminate beams. The system of equilibrium equations written in this form has only $n-m$ degrees of freedom. Procedures may therefore be used to eliminate m variables, but this may not be economical computationally for structures other than a simple one-span beam, (see Foulkes [2] for this discussion). In what follows, (8) will be treated as it stands. It should be noted that the constants $c_j \geq 0$ and, for simplicity, the coefficients a_{ij} may be identified with the coefficients of the equations

$$\text{(9)} \qquad \frac{M_i - M_{i-1}}{x_i - x_{i-1}} + \frac{M_i - M_{i+1}}{x_{i+1} - x_i} = p_i, \qquad i = 1, \ldots, m.$$

Two special cases of (8) will now be considered.

Case 1. φ is either linear or strictly convex; $f(M) = \max\{|M|, R_0\}$. The problem is that of a convex program with linear constraints:

$$\text{(10a)} \qquad \text{Minimize } C_1(R, M) = \sum_{j=1}^{n} c_j \varphi(R_j),$$

subject to

$$M_j \leq R_j, \qquad -M_j \leq R_j, \qquad R_0 \leq R_j \quad \text{for all } j;$$

$$\sum_{j=1}^{n} a_{ij} M_j = p_i \qquad \text{for all } i.$$

The dual problem of (10a) can be written as

$$\text{(10b)} \qquad \text{Maximize } U_1 = \sum_{j=1}^{n} \left\{ c_j\varphi(R_j) - R_j \frac{\partial C_1}{\partial R_j} + R_0\phi_j \right\} + \sum_{i=1}^{m} p_i u_i,$$

subject to

$$\sum_{i=1}^{m} a_{ij}u_i = \theta_j^+ - \theta_j^-, \quad \text{for all } j;$$

$$\theta_j^+ + \theta_j^- + \phi_j = \frac{\partial C_1}{\partial R_j}, \quad \text{for all } j;$$

$$\theta_j^+, \theta_j^-, \phi_j \geq 0, \quad \text{for all } j.$$

Since the duality theorem holds, the following conditions are satisfied at the optimum (from equations (1), (2)):

$$\text{(11a)} \qquad C_1(\bar{R}, \bar{M}) = U_1(\bar{R}, \bar{M}; \bar{\theta}, \bar{\phi}, \bar{u}),$$

$$\text{(11b)} \qquad \bar{\theta}_j^+(\bar{R}_j - \bar{M}_j) = \bar{\theta}_j^-(\bar{R}_j + \bar{M}_j) = \bar{\phi}_j(\bar{R}_j - R_0) = 0, \quad \text{for all } j.$$

If the variables u_i are interpreted as virtual displacements at x_i and the corresponding rotations (changes of curvature) are $\theta_j = \theta_j^+ - \theta_j^-$, then the first constraint equations of (10b) are seen to be the geometrical relationship between these variables by referring to (9). The second constraint equations of (10b) and (11b) will give

$$\text{(12a)} \qquad \operatorname{sgn} \bar{\theta}_j = \operatorname{sgn} \bar{M}_j,$$

$$\text{(12b)} \qquad \frac{\partial C_1(\bar{R}, \bar{M})}{\partial R_j} = \begin{cases} |\bar{\theta}_j| \text{ (with } \bar{\phi}_j = 0) & \text{for } |\bar{M}_j| = \bar{R}_j > R_0, \\ |\bar{\theta}_j| + \bar{\phi}_j & \text{for } |\bar{M}_j| = \bar{R}_j = R_0, \\ \bar{\phi}_j \text{ (with } \bar{\theta}_j = 0) & \text{for } |\bar{M}_j| < \bar{R}_j = R_0. \end{cases}$$

If the variables ϕ_j are interpreted as fictitious rotations for the resistance R_0, equations (12) give the necessary and sufficient conditions for the virtual deformation associated with the optimal design. In case $R_0 = 0$, the variables ϕ_j are no longer present. Equations (12) are then the optimal conditions obtained by Heyman [19] for the case where $C_1(R, M)$ is a linear function. In view of (12b), equation (11a) may be rewritten as

$$\text{(12c)} \qquad \begin{aligned} \sum_{j=1}^{n} c_j\varphi(\bar{R}_j) = C_1(\bar{R}, \bar{M}) &= U_1(\bar{R}, \bar{M}; \bar{\theta}, \bar{\phi}, \bar{u}) \\ &= \sum_{j=1}^{n} \left\{ c_j\varphi(\bar{R}_j) - \bar{R}_j \frac{\partial C_1}{\partial R_j} + R_0\bar{\phi}_j \right\} + \sum_{i=1}^{m} p_i\bar{u}_i \\ &= \sum_{j=1}^{n} \{ c_j\varphi(\bar{R}_j) - \bar{M}_j\bar{\theta}_j \} + \sum_{i=1}^{m} p_i\bar{u}_i. \end{aligned}$$

The importance of this relation is recognised by Marçal and Prager ([7], equation (15)) from the theory of non-linear elasticity. If C_1 is the total complementary energy of a structure for the statically admissible bending moments M_j, then $U_1 (= -E)$ is the negative value of the potential energy for the kinematically admissible deformation u_i, θ_j. The total complementary energy and the potential energy reach the same minimum when the deformation is "compatible" with the bending moments, i.e. when they are the elastic solution of the non-linear elastic problem with moment–curvature relation given by (12b).

Prager and Shield [8] have recently extended this result to two dimensional structures. The programming approach here suggests a method of numerical solution to their theory.

Case 2. $\varphi(R) = R^\alpha (0 < \alpha < 1); f(M) = |M|$. This is the minimum weight design problem considered by Prager [4] and Vargo [5]. The cost function is concave so that the duality theorem is no longer applicable. However, Theorems 1, 2 are still valid. The minimization problem (8) now becomes

$$\text{(13)} \qquad \text{Minimize } C_2(R, M) = \sum_{j=1}^{n} c_j R_j^\alpha,$$

subject to

$$M_j \le R_j, \quad -M_j \le R_j, \quad j = 1, \ldots, n;$$

$$\sum_{j=1}^{n} a_{ij} M_j = p_i, \qquad i = 1, \ldots, m.$$

By writing the Lagrangian function as

$$U_2 = \sum_{j=1}^{n} \{c_j R_j^\alpha + \theta_j^+ (M_j - R_j) + \theta_j^- (-M_j - R_j)\} + \sum_{i=1}^{m} u_i \left(p_i - \sum_{j=1}^{n} a_{ij} M_j \right),$$

the necessary conditions of optimum are given by (1), (2):

$$\text{(14a)} \qquad \sum_{i=1}^{m} a_{ij} \bar{u}_i = \bar{\theta}_j^+ - \bar{\theta}_j^- = \bar{\theta}_j, \qquad \bar{\theta}_j^+, \bar{\theta}_j^- \ge 0, \quad \text{for all } j;$$

$$\text{(14b)} \qquad \bar{\theta}_j^+ (\bar{R}_j - \bar{M}_j) = \bar{\theta}_j^- (\bar{R}_j + \bar{M}_j) = 0, \qquad \text{for all } j;$$

$$\text{(14c)} \qquad \alpha c_j \bar{R}_j^{\alpha - 1} = \bar{\theta}_j^+ + \bar{\theta}_j^- = |\bar{\theta}_j|, \qquad \text{for all } j.$$

The variables u_i, θ_j have the same physical meaning as in Case 1. By using (14), the sufficient condition (3) can be written as

$$\text{(15)} \qquad \sum_{j=1}^{n} \{c_j \bar{R}_j^\alpha - \bar{M}_j \bar{\theta}_j\} + \sum_{i=1}^{m} p_i \bar{u}_i \le \sum_{j=1}^{n} \{c_j R_j^\alpha - M_j \bar{\theta}_j\} + \sum_{i=1}^{m} p_i \bar{u}_i$$

for all R_j, M_j satisfy the constraints of (13). However, since $M_j, \bar{M}_j$ satisfy the equations of equilibrium, they are statically admissible bending moments. Moreover (14a) shows that $\bar{u}_i, \bar{\theta}_j$ are kinematically admissible deformations. So that, by the principle of virtual work, (15) reduces simply to

$$\sum_{j=1}^{n} c_j \bar{R}_j^\alpha \le \sum_{j=1}^{n} c_j R_j^\alpha,$$

which is explained by Prager [4] as "there must not be any admissible design of lesser weight admitting the failure mechanism $\bar{u}_i, \bar{\theta}_j$ satisfying equations (14)".

It is shown here that, without the convexity requirement on the cost function, equations (14) alone are not sufficient to give the optimum. However, further progress may still be made by using the following theorem (see Hadley [17], p. 93).

THEOREM 3. *If the constraint equations form a convex region which is closed and bounded from below then the minimum of a concave function, if it is finite, will be attended on an extreme point of the convex region.*

In case the constraints are all linear, there are only finite number of extreme points on the convex region. "Simplex" type methods which search for the minimum from one extreme point to the next can be used to obtain the solution. This fact is recognised by Vargo [5].

4. OPTIMAL DESIGN FOR MULTIPLE SETS OF LOADS

A structure may be required to carry different sets of loads $p^{(r)}(x)$ at different times. Shield [20] has given sufficient conditions for structures to be of minimum weight under multiple loading conditions. Gross and Prager [12] have considered the numerical solutions of an optimal beam subjected to a single moving load (multiple loading with $r \to \infty$) by the method of linear programming. Recently, Mayeda and Prager [21] studied the minimum weight design of beams for multiple loading by using the theory of [20]. These linear problems are well within the scope of linear programming as demonstrated by the work of Chan [15] of a similar nature.

A slightly general case of optimal design of beams will now be considered. The cost function is assumed to be convex and there are k sets ($r = 1, \ldots, k$) of loadings. In analogy with (8), the problem may be formulated as

$$\text{(16a)} \qquad \text{Minimize } C_3 = \sum_{j=1}^{n} c_j \varphi(R_j),$$

subject to

$$M_j^{(r)} \leq R_j, \qquad -M_j^{(r)} \leq R_j, \qquad R_0 \leq R_j, \qquad j = 1, \ldots, n; \quad r = 1, \ldots, k;$$

$$\sum_{j=1}^{n} a_{ij} M_j^{(r)} = p_i^{(r)}, \qquad i = 1, \ldots, m; \quad r = 1, \ldots, k.$$

The dual problem of (16a) is written as

$$\text{(16b)} \qquad \text{Maximize } U_3 = \sum_{j=1}^{n} \left\{ c_j \varphi(R_j) - c_j R_j \frac{\partial \varphi}{\partial R_j} + R_0 \phi_j \right\} + \sum_{i=1}^{m} \sum_{r=1}^{k} u_i^{(r)} p_i^{(r)},$$

subject to

$$\sum_{i=1}^{m} a_{ij} u_i^{(r)} = \theta_j^{(r+)} - \theta_j^{(r-)}, \qquad \text{for all } j, r;$$

$$c_j \frac{\partial \varphi}{\partial R_j} = \sum_{r=1}^{k} (\theta_j^{(r+)} + \theta_j^{(r-)}) + \phi_j, \qquad \text{for all } j;$$

$$\phi_j, \theta_j^{(r+)}, \theta_j^{(r-)} \geq 0, \qquad \text{for all } j, r.$$

Since the functions $\varphi(R_j)$ are assumed to be convex and the constraints of (16a) are all linear, the duality theorem holds. Similar interpretations as in case 1 can be made. $u_i^{(r)}$ and

$\theta_j^{(r)} = \theta_j^{(r+)} - \theta_j^{(r-)}$ are virtual deformations associated with the loads $p_i^{(r)}$. From (1), (2), the following conditions are satisfied at the optimum:

$$C_3(\bar{R}, \bar{M}) = U_3(\bar{R}, \bar{M}; \bar{\theta}, \bar{\phi}, \bar{u}), \tag{17a}$$

$$\bar{\theta}_j^{(r+)}(\bar{R}_j - \bar{M}_j^{(r)}) = \bar{\theta}_j^{(r-)}(\bar{R}_j + \bar{M}_j^{(r)}) = \bar{\phi}_j(\bar{R}_j - R_0) = 0 \qquad \text{for all } j, r. \tag{17b}$$

In the linear case, interpretation of these results leads to the sufficient condition of optimum as given by Mayeda and Prager [21]. However, this condition on the associated deformations $u_i^{(r)}$ and $\theta_j^{(r)}$ is now shown to be also necessary for the optimum. The function $U_3(R, M; \theta, \phi, u)$ of (16b) has the same physical meaning as that of (10b), which shows that the design method of Prager and Shield [8] can be generalized to the case of multiple loads.

5. MINIMUM WEIGHT DESIGN OF SANDWICH PLATES

In this section, it is intended to apply the programming method to problems with a non-linear yield condition such as the von Mises yield condition. However, the duality theorem of section 2 can only be used when, for example, the cost function is linear. For sandwich plates under rotational symmetry conditions, Freiberger and Takinalp [22] have obtained the criterion for minimum weight by using the calculus of variations. By following their notations, the design problem of a circular sandwich plate simply supported at the edge $r = R$ can be formulated as

$$\text{Minimize } C_4 = 2\pi \int_0^R rh \, \mathrm{d}r,$$

subject to

$$\sigma_0 H h = F(M, N) = (M^2 + N^2 - MN)^{\frac{1}{2}},$$

$$r\frac{\mathrm{d}M}{\mathrm{d}r} + M - N = -\int_0^r rp(r)\,\mathrm{d}r.$$

By dividing the region into equal intervals $0 < r_1 < r_2 < \ldots < r_n = R$, $r_{i+1} - r_i = \Delta r$, the following programming problem is obtained:

$$\text{Minimize } C_4 = \frac{2\pi}{\sigma_0 H} \sum_{i=1}^{n} r_i \Delta r F(M_i, N_i), \tag{18a}$$

subject to

$$\left(1 - \frac{r_i}{\Delta r}\right) M_i + \frac{r_i}{\Delta r} M_{i+1} - N_i = p_i = -\sum_{j=1}^{i} r_j p(r_j) \Delta r, \qquad i = 1, \ldots, n-1.$$

Since the functions $F(M_i, N_i)$ are convex and the constraints are linear, the duality theorem holds for (18a) and its dual problem which may be written as

$$\text{Maximize } U_4 = \sum_{i=1}^{n-1} p_i y_i, \tag{18b}$$

subject to

$$\frac{2\pi}{\sigma_0 H} r_i \Delta r \frac{\partial F}{\partial N_i} = -y_i, \qquad i = 1, \ldots, n-1;$$

$$\frac{2\pi}{\sigma_0 H} r_i \Delta r \frac{\partial F}{\partial M_i} = \frac{r_{i-1}}{\Delta r} y_{i-1} + \left(1 - \frac{r_i}{\Delta r}\right) y_i, \qquad i = 1, \ldots, n-1; y_0 = 0.$$

By letting $\sigma_0 H y_i = 2\pi(v_{i+1} - v_i)$, the above equations can be rewritten as

(19a) $$U_4 = \sum_{i=1}^{n-1} p_i y_i = \frac{2\pi}{\sigma_0 H} \sum_{i=1}^{n-1} r_i v_i p(r_i) \Delta r \simeq \frac{2\pi}{\sigma_0 H} \int_0^R r v p(r)\, \mathrm{d}r,$$

(19b) $$\frac{\partial F}{\partial N_i} = \frac{v_i - v_{i+1}}{r_i \Delta r} \simeq -\frac{v_i'}{r_i},$$

(19c) $$\frac{\partial F}{\partial M_i} = \frac{(r_i - \Delta r)}{r_i} \frac{(2v_i - v_{i+1} - v_{i-1})}{(\Delta r)^2} \simeq -v_i'';$$

where primes denote differentiation with respect to r. The variables v_i are the rates of deflection of the plate. (19a) shows that U_4, and hence the weight of the design, is proportional to the total work done by the external forces. Equations (19b, c) are merely the deflection–curvature relation and the associated flow rule (see [22], equations (1), (5)), which combine to give, in the limit, the criterion for optimum (see [22], equation (12)):

(19d) $$\frac{\partial F}{\partial M} = \frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\partial F}{\partial N}\right).$$

6. CONCLUSIONS

The paper demonstrates that, by using the methods of mathematical programming, optimal design results which are previously derived from quite different approaches can be obtained. In case of convex programs, necessary and sufficient conditions for optimum are obtained. This suggests immediately that programming methods can be very useful in the formulation of problems. Deeper insight into the design problem may be achieved by extending the present results to the case of non-linear constraints.

Even if analytical solutions are not available, mathematical programming can always obtain numerical solutions. Experience in the application of linear programming shows very good agreement between theory and computation. As for the non-linear cases, more computational studies have to be made before any conclusions can be drawn. However, certain observations may be made. Rosen's method [23] or Zoutendijk's methods [24] seem very suitable for solving problems presented here. It should be noted also that a feasible solution (not necessarily the optimum) of the minimization problem will give an upper bound and a feasible solution of its dual problem will give a lower bound to the cost. Such bounds may provide a sound basis for evaluating the merit of a practical design.

Finally, it is worth noticing that the techniques of mathematical programming have long been used in the adjacent fields of study such as limit analysis and optimum elastic design of structures. In the former case, the works of Dorn and Greenberg [25] and Charnes *et al.* [26] make significant contributions to both the fields of mathematical programming and plastic limit analysis. These results can, in effect, be applied directly to the problem of plastic limit design. In the latter case, extensive studies have been made by an increasing number of authors. To mention just a few, for example L. A. Schmit Jr., R. L. Fox, R. Razani, G. G. Pope, T. P. Kicher and etc. The Fourth Conference on Electronic Computation organized by the Structural Division of the ASCE (Published in *Proc. Am. Soc. civ. Engrs* **92**, ST6, Dec. 1966) did reflect some trends of research along this direction and the interested readers may obtain references from this publication. It seems that mathematical programming can also provide a common basis for comparing the results of different structural design methods.

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Абстракт—Работа имеет целью определить задачи оптимального пластического расчета конструкций, с точки зрения, математического программирования. Показываются, что настоящие методы оптимального расчета соответствуют разным формулировкам математических программ. Двойственные теоремы математического программирования могут быть использованы для определения конечных и достаточных критериев оптимума. Показано, также, программирование как метод численного расчета.